JOURNAL OF DIFFERENTIAL EQUATIONS 22, 305–313 (1976)

Existence Theorems for Nonlinear Noncoercive Operator Equations and Nonlinear Elliptic Boundary Value Problems

CHAITAN P. GUPTA

Northern Illinois University, Department of Mathematics, De Kalb, Illinois 60115

AND

PETER HESS

University of Zürich, Mathematics Institute, Freiestrasse 36, 8032 Zürich, Switzerland

Received May 19, 1975

INTRODUCTION

The purpose of this paper is to study the solvability of a nonlinear functional equation $f \in (A + B)u$, where A is a maximal monotone (possibly multivalued) mapping from a reflexive Banach space X to its dual X^* and $B: X \to X^*$ satisfies some kind of sign condition. The main feature of the present method is that no asymptotic hypothesis (such as coerciveness, semicoerciveness, asymptotic oddness or homogeneity) is imposed on $A + B$.

In Section 1 we state and prove the abstract result (Theorem 1). In Section 2 we apply Theorem 1 to the discussion of solvability of nonlinear second-order boundary value problems of Neumann type. Such problems were previously considered by Schatzman [6] and Hess [5] by different methods.

This research was stimulated by some recent results on the range of the sum of maximal monotone operators announced by Brézis [3].

1

Let X be a real reflexive Banach space and X^* its dual space. We denote by (w, u) the duality pairing between the elements $w \in X^*$ and $u \in X$. Let $A: X \to 2^{X^*}$ be a given mapping. Its *effective domain* $D(A)$ is the subset of X defined by $D(A) = \{u \in X: Au \neq \varnothing\}$, its *range* $R(A)$ the subset of X^* defined by $R(A) = \bigcup\{Au: u \in D(A)\}$, and its *graph* $G(A)$ the subset of

$X \times X^*$ given by $G(A) = \{[u, w]: u \in D(A),\ w \in Au\}$. The mapping A is said to be ***monotone*** if its graph $G(A)$ is a monotone subset of $X \times X^*$ in the sense that $(w_1 - w_2, u_1 - u_2) \geqslant 0$ for $[u_i, w_i] \in G(A)$, $i = 1, 2$. Further A is ***maximal monotone*** if $G(A)$ is not a proper subset of any other monotone subset of $X \times X^*$. The mapping A is said to be ***trimonotone*** if for any triple of elements $u_i \in D(A)$ and $w_i \in Au_i$ $(i = 1, 2, 3)$ we have $(w_1, u_1 - u_2) + (w_2, u_2 - u_3) + (w_3, u_3 - u_1) \geqslant 0$. A singlevalued mapping $T: X \to X^*$ is said to be *bounded* if it maps bounded subsets of X into bounded subsets of X^*, and *compact* if it maps bounded subsets of X into relatively compact subsets of X^*. Further $T: X \to X^*$ is said to be *demicontinuous* if it is continuous from X to X^* endowed with the weak topology. For any subset G we denote by *Int* G its interior.

DEFINITION. Let $A: X \to 2^{X^*}$ be a given mapping. We say that A is *boundedly-inversly-compact* if for any pair of bounded subsets G and G^* of X and X^*, respectively, the subset $G \cap A^{-1}(G^*)$ is relatively compact in X.

For example, if $K: X^* \to X$ is compact, then the mapping $K^{-1}: X \to 2^{X^*}$ is boundedly-inversely-compact.

We now state our main abstract result.

THEOREM 1. *Let X be a real reflexive Banach space. Let $A: X \to 2^{X^*}$ be a monotone mapping, $B_1 : X \to 2^{X^*}$ a trimonotone mapping such that the following hold*:

(i) $D(A) \subset D(B_1)$

(ii) $0 \in (A + B_1)(0)$

(iii) $A + B_1 : X \to 2^{X^*}$ *is maximal monotone and boundedly-inversely-compact.*

Further let $B_2 : X \to X^*$ be a demicontinuous mapping satisfying the condition:

$$\text{for every } k \geqslant 0 \text{ there exists a constant } c(k) \text{ such that } (B_2 u, u) \geqslant k \, \| B_2 u \| - c(k) \text{ for all } u \in X. \tag{1.1}$$

Then $w \in \operatorname{Int}(R(A) + R(B_1))$ implies that $w \in \operatorname{Int} R(A + B_1 + B_2)$.

Remark 1. When $B_2 \equiv 0$ we do not need to assume that the mapping $A + B_1$ is boundedly-inversely-compact. In that case the result is essentially due to Brézis [3].

Remark 2. As a consequence of condition (1.1) the mapping B_2 is bounded.

For the proof of Theorem 1 we need two auxiliary results. The first statement immediately follows from the Leray–Schauder principle.

PROPOSITION 1. *Let X be a real Banach space and $K: X^* \to X$ a (nonlinear) compact demicontinuous monotone mapping with $K(0) = 0$. Let $F: X \to X^*$ be another (nonlinear) bounded demicontinuous mapping. Suppose that for some constant $\rho > 0$,*

$$(Fu, u) > 0 \tag{1.2}$$

for all $u \in X$ with $\| u \| = \rho$. Then the equation

$$u - K(-Fu) = 0$$

admits a solution u with $\| u \| < \rho$.

Using a result of Asplund [1], we shall assume in future that the reflexive Banach space X is endowed with a norm such that both X and X^* are strictly convex. Accordingly the duality mapping J given by $Ju = \{v \in X^*: (v, u) = \| u \|^2, \| v \| = \| u \|\}$ for $u \in X$, is everywhere defined, single-valued, bounded and demicontinuous (see e.g. [4]).

PROPOSITION 2. *Let the Banach space X be real and reflexive. Let $T: X \to 2^{X^*}$ be a maximal monotone, boundedly-inversely-compact mapping with $0 \in T(0)$. Further let $B: X \to X^*$ be a bounded demicontinuous mapping such that*

$$(Bu, u) \geqslant -c \| u \| - d \tag{1.3}$$

for all $u \in X$, with some constants c, d. Then for each $\epsilon > 0$ the mapping $T + B + \epsilon J: X \to 2^{X^*}$ is surjective.

Proof. It is well known that the mapping $(T + (\epsilon/2)J)^{-1}$ is everywhere defined, single-valued, bounded and demicontinuous. We also note that it is compact. To see this, let $\{u_n\}$ be a bounded sequence in X^* and let $x_n = (T + (\epsilon/2)J)^{-1}u_n$. Thus $u_n = w_n + (\epsilon/2) Jx_n$, with $w_n \in Tx_n$. Since $(T + (\epsilon/2)J)^{-1}$ is a bounded mapping, the sequence $\{x_n\}$ is bounded in X. Accordingly $\{w_n\}$ is a bounded sequence in X^*. It then follows from the boundedly-inversely-compact nature of T that there exists a subsequence $\{x_{n_i}\}$ which is convergent in X. This shows that $(T + (\epsilon/2)J)^{-1}$ maps bounded subsets of X^* into relatively compact subsets of X. Also $(T + (\epsilon/2)J)^{-1}(0)=0$.

Next we see that the equation $w \in Tu + Bu + \epsilon Ju$ is equivalent to the equation $u = (T + (\epsilon/2)J)^{-1} [w - (B + (\epsilon/2)J)u]$. It now follows immediately from Proposition 1 that this second equation has a solution.

Proof of Theorem 1. Let $w \in \operatorname{Int}(R(A) + R(B_1))$. By Proposition 2 (applied with $T = A + B_1$, $B = B_2$), there exists for each $\epsilon > 0$ an element $u_\epsilon \in X$ such that

$$w \in Au_\epsilon + B_1u_\epsilon + B_2u_\epsilon + \epsilon Ju_\epsilon .$$

Since $0 \in (A + B_1)(0)$, $(Ju_\epsilon, u_\epsilon) = \| u_\epsilon \|^2$ and B_2 satisfies condition (1.1), we see that there exists a constant C such that

$$\epsilon \| u_\epsilon \| \leqslant C, \qquad \forall \epsilon > 0. \tag{1.4}$$

Now, $w \in \operatorname{Int}(R(A) + R(B_1))$ implies the existence of a constant $\rho > 0$ such that for $h \in X^*$ with $\| h \| \leqslant \rho$ we have $w - h \in R(A) + R(B_1)$. We assert that to each such $h \in X^*$ there is a constant C_h with

$$(h, u_\epsilon) \geqslant C_h, \qquad \forall \epsilon > 0. \tag{1.5}$$

Indeed, let $\xi_\epsilon \in Au_\epsilon$, $\eta_\epsilon \in B_1 u_\epsilon$ be such that

$$w = \xi_\epsilon + \eta_\epsilon + B_2 u_\epsilon + \epsilon J u_\epsilon, \tag{1.6}$$

and let $y \in D(A)$, $z \in D(B_1)$ be such that

$$w - h = \xi + \eta, \tag{1.7}$$

where $\xi \in Ay$, $\eta \in B_1 z$. Subtracting (1.7) from (1.6) we get

$$h = \xi_\epsilon - \xi + \eta_\epsilon - \eta + B_2 u_\epsilon + \epsilon J u_\epsilon .$$

This gives

$$\begin{aligned} (h, u_\epsilon - y) &= (\xi_\epsilon - \xi, u_\epsilon - y) + (\eta_\epsilon - \eta, u_\epsilon - y) \\ &\quad + (B_2 u_\epsilon + \epsilon J u_\epsilon, u_\epsilon - y) \\ &\geqslant (\eta_\epsilon - \eta, u_\epsilon - y) + (B_2 u_\epsilon, u_\epsilon) \\ &\quad - \| B_2 u_\epsilon \| \cdot \| y \| - \epsilon \| u_\epsilon \| \cdot \| y \| \\ &\geqslant (\eta_\epsilon - \eta, u_\epsilon - y) \text{—constant (depending on } h), \end{aligned} \tag{1.8}$$

in view of condition (1.4) and (1.1) applied with $k = \| y \|$ to the terms $(B_2 u_\epsilon, u_\epsilon) - \| B_2 u_\epsilon \| \cdot \| y \|$. Since B_1 is trimonotone and $D(A) \subset D(B_1)$, we see by taking $\zeta \in B_1 y$ that

$$\begin{aligned} 0 &\leqslant (\eta_\epsilon, u_\epsilon - y) + (\zeta, y - z) + (\eta, z - u_\epsilon) \\ &= (\eta_\epsilon, u_\epsilon - y) + (\zeta, y - z) + (\eta, z - y) + (\eta, y - u_\epsilon) \\ &= (\eta_\epsilon - \eta, u_\epsilon - y) + (\zeta - \eta, y - z). \end{aligned}$$

This shows that $(\eta_\epsilon - \eta, u_\epsilon - y) \geqslant$ constant (depending on h). The assertion (1.5) now follows from (1.8).

Applying the Principle of Uniform Boundedness we see that for some constant M,

$$\| u_\epsilon \| \leqslant M, \qquad \forall \epsilon > 0. \tag{1.9}$$

Since $w = \xi_\epsilon + \eta_\epsilon + B_2 u_\epsilon + \epsilon J u_\epsilon$, we have that $\xi_\epsilon + \eta_\epsilon + B_2 u_\epsilon \to w$ as $\epsilon \to 0$. Since the sets $\{u_\epsilon\}$ and $\{B_2 u_\epsilon\}$ are bounded, we conclude that the set $\{\xi_\epsilon + \eta_\epsilon\}$ is also bounded. By the boundedly-inverse compactness of $A + B_1$, there exists a sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$, and an element $u \in X$ such that $u_{\epsilon_n} \to u$ in X. So we have $B_2 u_{\epsilon_n} \rightharpoonup B_2 u$ (weakly) in X^* and thus $\xi_{\epsilon_n} + \eta_{\epsilon_n} \rightharpoonup w - B_2 u$ (weakly) in X^*. Since $A + B_1$ is maximal monotone, $w - B_2 u \in (A + B_1)u$, i.e., $w \in (A + B_1 + B_2)u$. Hence $w \in \operatorname{Int}(R(A) + R(B_1))$ implies that $w \in R(A + B_1 + B_2)$, which in turn implies that $\operatorname{Int}(R(A) + R(B_1)) \subset \operatorname{Int} R(A + B_1 + B_2)$. This completes the proof of Theorem 1.

2

Let Ω be a bounded domain in an Euclidean space $\mathbb{R}^N$ ($N \geqslant 1$) with smooth boundary Γ. Let β be a maximal monotone graph in $\mathbb{R}^2$ with $0 \in \beta(0)$. Further let

$$\mathscr{A} = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

denote a uniformly elliptic differential expression with realvalued coefficients $a_{ij} \in C^1(\bar{\Omega})$ $(i, j = 1,..., N)$. To $\mathscr{A}$ we associate the nonlinear operator $A: L^2(\Omega) \to 2^{L^2(\Omega)}$ defined by

$$D(A) = \left(u \in H^2(\Omega) : -\frac{\partial u}{\partial n_a} \in \beta(u) \text{ a.e. on } \Gamma\right),$$

$$Au = \mathscr{A}u \qquad \text{for} \quad u \in D(A).$$

Here $\partial u/\partial n_a$ denotes the outward conormal derivative given by

$$\frac{\partial u}{\partial n_a} = \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i),$$

where n is the outward normal to Γ. It is known from a result of Brézis [2] that A is maximal monotone. This implies in particular that for any given $f \in L^2(\Omega)$ and $\epsilon > 0$ the equation $f = Au + \epsilon u$ has a unique solution u.

For any $t \in \mathbb{R}$ let $\beta^0(t) \in \beta(t)$ be the element with least absolute value, where we set $\beta^0(t) = \pm\infty$ in case $\beta(t) = \varnothing$ (and $t \gtrless 0$, respectively). Finally let

$$\beta_\pm = \lim_{t\to\pm\infty} \beta^0(t) \quad \text{(in the extended sense).}$$

PROPOSITION 3. *Let $f \in L^2(\Omega)$ be such that*

$$\beta_- < \frac{1}{\operatorname{meas}(\Gamma)} \int_\Omega f\, dx < \beta_+ . \tag{2.1}$$

Then $f \in \operatorname{Int} R(A)$.

This result is originally due to Schatzman [6]. We present here a direct and simpler proof of it. Prior to that we introduce some further notation. For a function $u \in H^1(\Omega)$ we let u/Γ denote its trace.

Proof of Proposition 3. Let $f \in L^2(\Omega)$ be such that (2.1) holds. For each $n \in \mathbb{N}$ there is a $u_n \in D(A)$ with

$$Au_n + (1/n)u_n = f. \tag{2.2}$$

We first assert that $\| u_n \| \leqslant$ const. $\forall n$ (where $\| \cdot \|$ means the norm in $L^2(\Omega)$). Indeed, suppose to the contrary that $\| u_n \| \to \infty$ as $n \to \infty$, and let $v_n = \| u_n \|^{-1} u_n$. Further let $b_n = -\partial u_n/\partial n_a$. Taking the inner product of (2.2) with u_n and applying Green's formula we obtain

$$\int_\Omega \sum a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial u_n}{\partial x_i}\, dx + \int_\Gamma b_n u_n/\Gamma\, d\Gamma + \frac{1}{n} \| u_n \|^2 = (f, u_n)$$

$\forall n$, and dividing this equation by $\| u_n \|^2$ we get

$$\int_\Omega \sum a_{ij} \frac{\partial v_n}{\partial x_j} \frac{\partial v_n}{\partial x_i}\, dx + \frac{1}{n} \| v_n \|^2 + \| u_n \|^{-1} \left\{ \int_\Gamma b_n v_n/\Gamma\, d\Gamma - (f, v_n) \right\} = 0. \tag{2.3}$$

Since $\int_\Gamma b_n v_n/\Gamma\, d\Gamma \geqslant 0$, we conclude from (2.3) and the uniform ellipticity of $\mathscr{A}$ that grad $v_n \to 0$ in $(L^2(\Omega))^N$. Thus, after passage to a suitable subsequence,

$$v_n \to v = \text{const} \neq 0 \quad \text{in} \quad H^1(\Omega).$$

Suppose first that $v > 0$. By (2.3),

$$\liminf \int_\Gamma b_n v_n/\Gamma\, d\Gamma \leqslant (f, v),$$

while

$$b_n v_n/\Gamma \geqslant \beta^0(u_n/\Gamma)\, v_n/\Gamma \qquad \text{a.e. on } \Gamma$$

and (for a further subsequence)

$$u_n/\Gamma \to +\infty \qquad \text{a.e. on } \Gamma.$$

By the Fatou lemma and the definition of β_+ it follows that

$$\beta_+ v \operatorname{meas}(\Gamma) = \int_\Gamma \beta_+ v/\Gamma \, d\Gamma \leqslant (f, v) = v \int_\Omega f \, dx,$$

contradicting (2.1). Similarly $v < 0$ leads to a contradiction.

Hence the assertion that $\| u_n \| \leqslant$ const follows. It is now an immediate consequence of the maximal monotonicity of A that $f \in R(A)$. Because of the strict inequality signs in (2.1) we even get $f \in \operatorname{Int} R(A)$.

Let now $g: \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following:

(i) for some constants a, b,

$$| g(t) | \leqslant a + b \, | t |, \qquad \forall t \in \mathbb{R};$$

(ii) there is a $T \geqslant 0$ such that

$$g(t)t \geqslant 0 \qquad \forall \, | t | \geqslant T.$$

Further, let

$$g_+ = \liminf_{t \to +\infty} g(t), \qquad g_- = \limsup_{t \to -\infty} g(t).$$

THEOREM 2. *Let $f \in L^2(\Omega)$ be such that*

$$\operatorname{meas}(\Gamma) \, \beta_- + \operatorname{meas}(\Omega) \, g_- < \int_\Omega f \, dx < \operatorname{meas}(\Gamma) \, \beta_+ + \operatorname{meas}(\Omega) \, g_+ . \tag{2.4}$$

Then the equation

$$\begin{aligned} \mathscr{A}u + g(u) &= f \qquad \text{a.e. in } \Omega, \\ -\partial u/\partial n_a &\in \beta(u) \qquad \text{a.e. on } \Gamma, \end{aligned} \tag{2.5}$$

has a solution $u \in H^2(\Omega)$.

Proof. Let $A: L^2(\Omega) \to 2^{L^2(\Omega)}$ be the maximal monotone operator as defined above. We decompose g into a sum $g = g_1 + g_2$ of continuous functions such that

$$\begin{gathered} g_1 \text{ is monotonically increasing, } g_1(0) = 0, \\ \lim_{t \to \pm\infty} g_1(t) = g_\pm ; \qquad g_2(t)t \geqslant 0 \quad \forall \, | t | \geqslant T, \\ | g_i(t) | \leqslant c + d \, | t | \quad (i = 1, 2), \qquad \forall t \in \mathbb{R} \text{ and some constants } c, d. \end{gathered} \tag{2.6}$$

Let $B_i : L^2(\Omega) \to L^2(\Omega)$ be given by $(B_i u)(x) = g_i(u(x))$ $(x \in \Omega, i = 1, 2)$. Then B_1 is a continuous monotone potential mapping. So B_1 is trimonotone.

Further B_2 satisfies condition (1.1) of Theorem 1. Indeed, for $u \in L^2(\Omega)$ we have

$$\begin{aligned}(B_2 u, u) &= \int_\Omega g_2(u(x))\, u(x)\, dx \\ &= \int_{\{x: |u(x)| \leqslant T\}} g_2(u) u\, dx + \int_{\{x: |u(x)| > T\}} g_2(u) u\, dx \\ &= I_1 + I_2 \quad \text{(say)}.\end{aligned}$$

Now $| I_1 | \leqslant$ const, in view of (2.6). Also,

$$\begin{aligned}I_2 &= \int_{\{x: |u(x)| > T\}} g_2(u) u\, dx = \int_{\{x: |u(x)| > T\}} | g_2(u) |\, | u |\, dx \\ &\geqslant \frac{1}{d} \int_{\{x: |u(x)| > T\}} [| g_2(u) |^2 - c\, | g_2(u) |]\, dx \\ &\geqslant (1/d) \| B_2 u \|^2 - (c/d)(\text{meas}(\Omega))^{1/2} \| B_2 u \| - \text{const}.\end{aligned}$$

Since for every $\epsilon > 0$ there exists a $C(\epsilon) > 0$ such that $p \leqslant \epsilon p^2 + C(\epsilon)$ $\forall p \in \mathbb{R}$, we see from above that for every $k \geqslant 0$ there is a constant $c'(k)$ such that

$$I_2 \geqslant k \| B_2 u \| - c'(k).$$

From this we immediately obtain (1.1).

We now show that the maximal monotone mapping $A + B_1$ from $L^2(\Omega)$ to $2^{L^2(\Omega)}$ is boundedly-inversely-compact. Indeed, let $\{u_n\}$ be a bounded sequence in $L^2(\Omega)$ with $\{w_n \in (A + B_1) u_n\}$ also bounded. We want to conclude that there is a subsequence of $\{u_n\}$ which is convergent in $L^2(\Omega)$. This follows from the Sobolev embedding theorem and the inequalities

$$\begin{aligned}\alpha \int_\Omega \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^2 dx &\leqslant \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial u_n}{\partial x_i}\, dx + (B_1 u_n, u_n) \\ &\leqslant (w_n, u_n) \qquad (\alpha > 0).\end{aligned}$$

To prove that Eq. (2.5) (or equivalently the equation: $(A + B_1 + B_2) u \ni f$) has a solution, it suffices by Theorem 1 to show that $f \in \text{Int}(R(A) + R(B_1))$. Two cases arise:

Case 1. $\beta_- < \beta_+$. In this case we can write

$$\int_\Omega f\, dx = \lambda\, \text{meas}(\Gamma) + \mu\, \text{meas}(\Omega),$$

where λ and μ are such that $\beta_- < \lambda < \beta_+$ and $g_- \leqslant \mu \leqslant g_+$. Writing $f = (f - \mu) + \mu$, we see that $\mu \in R(B_1)$ and $f - \mu \in \operatorname{Int} R(A)$ in view of Proposition 3, since

$$\beta_- < \frac{1}{\operatorname{meas}(\Gamma)} \int_\Omega (f - \mu)\, dx = \lambda < \beta_+ .$$

So $f \in \operatorname{Int}(R(A) + R(B_1))$, and the result follows.

Case 2. $\beta_- = \beta_+ = 0$. In this case condition (2.4) reduces to

$$g_- < \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega f\, dx < g_+ . \tag{2.4'}$$

Writing

$$k_f = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega f\, dx$$

we see that $(f - k_f) \in R(A)$ in view of the well-known existence theory for the linear Neumann boundary value problem. Moreover $k_f \in R(B_1)$ by (2.4′). Applying the same decomposition to small perturbations of f, we conclude that $f \in \operatorname{Int}(R(A) + R(B_1))$. This completes the proof of Theorem 2.

References

1. E. Asplund, Averaged norms, *Israel J. Math.* **5** (1957), 227–233.
2. H. Brézis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, *in* "Contributions to Nonlinear Functional Analysis" (E. H. Zarantonello, Ed.), pp. 101–156, Academic Press, New York, 1971.
3. H. Brézis, Monotone operators, non linear semi-groups and applications, *in* "Proc. Int. Congress Math.," Vancouver, 1974.
4. D. G. de Figueirido, Topics in nonlinear functional analysis, *in* "University of Maryland Lecture Notes," No. 48, 1967.
5. P. Hess, On semi-coercive nonlinear problems, *Indiana Univ. Math. J.* **23** (1974), 645–654.
6. M. Schatzman, Problèmes aux limites non linéaires, non coercifs. *Ann. Scuola Norm. Sup. Pisa* **27** (1973), 641–686.
7. H. Brézis and A. Haraux, Image d'une somme d'opérateurs monotones et applications, *Israel J. Math.* **23** (1976), 165–186.